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MODULI OF STABILITY OF TWO-DIMENSIONAL DIFFEOMORPHISMS

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INTRODUCTION

IN THIS paper we consider diffeomorphisms on a compact two-dimensional manifold having a pair of hyperbolic periodic points such that the unstable manifold of one of them is tangent to the stable manifold of the other along an orbit. They occur naturally in Bifurcation Theory: there is an open set of arcs, starting at a Morse–Smale diffeomorphism, whose first bifurcation point is such a diffeomorphism[3].

It is well known that a diffeomorphism f which exhibits an orbit of tangency between stable and unstable manifolds of hyperbolic periodic points cannot be structurally stable[7]. In fact, this situation gives rise to interesting invariants of topological equivalence, as pointed out in [4], which implies the existence of an uncountable number of different topological equivalence classes in any small neighborhood of f. However, it might be possible to parametrize all these equivalence classes with finitely many real parameters which would provide a pretty good description of the diffeomorphisms near f. In this case we say that the modulus of stability of f is finite and equal to the minimum number of parameters. Here we will prove that in some cases the modulus of stability is finite and in other cases it is infinite and that both situations occur for diffeomorphisms which are first bifurcation points for an open set of arcs starting at Morse–Smale diffeomorphisms.

The existence of those tangencies is not a persistent phenomenum in the space of all diffeomorphisms. However, if we restrict our perturbations to a specific subspace it may become persistent. This is the case when we consider the set of G-equivariant diffeomorphisms, where G is a finite group acting on the manifold. The action of G induces a partition of the manifold in submanifolds whose points have the same orbit type and this partition is let invariant by any equivariant diffeomorphism f. If f has two hyperbolic periodic points p_1 and p_2 such that the stable manifold of p_1 intersects the unstable manifold of p_2 and both are contained in the same one-dimensional submanifold of the partition then the same happens for any equivariant diffeomorphism near f. We prove here that, given any integer k, there is an open set of equivariant diffeomorphisms whose modulus of stability is exactly k.

§1. STATEMENT OF RESULTS

Let $\mathrm{Diff}^r(M)$ be the set of C^r, $2 \leq r \leq \infty$, diffeomorphisms on a compact C^∞ manifold M, endowed with C^r topology. The diffeomorphisms f and g are topologically equivalent if there is a homeomorphism h, called a conjugacy between f and g, such that $hf = gh$. This is clearly an equivalence relation and the equivalence classes are called conjugacy classes. If there is a neighborhood of f contained in its conjugacy class we say that f is structurally stable. If a small neighborhood of f intersects only finitely many conjugacy classes then the modulus of stability of f is equal to 0. In particular, a structurally stable diffeomorphism has zero modulus of stability.

Let p be a periodic point of f of period π. Then p is hyperbolic if the eigenvalues of $df^{\pi}(p)$ have absolute value different from one. If an eigenvalue has absolute value less than one and the other bigger than one we say that p is a saddle. The stable manifold of p, $W^s(p)$, is the set of points $x \in M$ such that $f^{n\pi}(x)$ converges to p as $n \to \infty$. The stable manifold of the orbit of p, $W^s(0(p))$, is the union of $W^s(q)$ for q in the orbit of p. Similarly, the unstable manifold of p, $W^u(p)$, is the set of points $x \in M$ such that $f^{-n\pi}(x)$ converges to p as $n \to \infty$. $W^s(p)$ and $W^u(p)$ are C^r immersed submanifolds of M which are transversal to each other at the point p.

Let p and q be hyperbolic periodic points of f. We say that $z \in W^s(p) \cap W^u(q)$ is a point of quasi-transversal intersection if $W^s(p)$ is tangent to $W^u(q)$ at z and the contact between them is parabolic. The same happens for any point in the orbit of z and it is called an orbit of quasi-transversal intersection.

In §1 we prove the following theorem.

THEOREM B. *Let $f \in \mathrm{Diff}^r(M)$. If there are k orbits of tangency between stable and that $W^u(p_1)$ has a tangency with $W^s(p_2)$. If there are infinitely many orbits in $W^s(0(p_1))$ belonging to unstable manifolds of periodic saddle points of f then the modulus of stability of f is infinite.*

In the same section we define a new conjugacy invariant which gives rise to the following theorem.

THEOREM B. *Let $f \in \mathrm{Diff}^r(M)$. If there are k orbitals of tangency between stable and unstable manifolds of hyperbolic saddle orbits of f then the modulus of stability of f is at least k.*

Recall that a cycle of periodic points of f is a sequence $p_1, p_2, \ldots, p_n = p_1$, of hyperbolic periodic points such that $W^u(p_i) \cap W^s(p_{i+1}) \neq \phi$ for $i = 1, \ldots, n-1$.

Let $U_k \subset \mathrm{Diff}^{\infty}(M)$ be the set of diffeomorphisms f satisfying the following properties:

(1) The set of non-wandering points of f is finite and hyperbolic;

(2) f has no cycles;

(3) There are only k orbits of tangency between stable and unstable manifolds of periodic points of f and these are orbits of quasi-transversal intersection;

(4) If p_1 and p_2 are periodic points of f such that $W^u(0(p_1))$ is not transversal to $W^s(0(p_2))$ then there is no saddle point of f whose unstable manifold intersects $W^s(0(p_1))$ or whose stable manifold intersects $W^u(0(p_2))$;

(5) If p is a periodic point of f of period π then if g is any diffeomorphism near f, g^{π} is C^2-linearizable in a neighborhood of p.

We observe that from [8], the set of diffeomorphisms satisfying condition (5) is open and dense in the set of diffeomorphisms satisfying conditions (1)–(4).

Section 3 is dedicated to the proof of the theorem below.

THEOREM C. *If $f \in U_k$ then the modulus of stability of f is equal to k.*

We notice that the set U_1 defined above is open in the boundary of the set of Morse–Smale diffeomorphisms. On the other hand, let us consider the set υ of diffeomosphisms satisfying conditions (1)–(3) above, for $k = 1$, and also the following property: if $W^u(0(p_1))$ is not transversal to $W^s(0(p_2))$ then there are saddle points q_1, q_2 such that $W^u(0(q_1)) \cap W^s(0(p_1)) \neq \phi$ and $W^u(0(q_2)) \cap W^s(0(q_1)) \neq \phi$. Then υ is also open in the boundary of the set of Morse–Smale diffeomorphisms and, by

Theorem A, any $f \in \upsilon$ has modulus of stability infinite. After the completion of this paper I realized upon remarks of S. Newhouse, C. Pugh and the referee that Theorem C can be extend to C^2 diffeomorphisms satisfying properties (1)–(4).

In §4 we deal with a situation which has codimension infinite in the set of all diffeomorphisms. We consider the set $\mathscr{m}$ of C^∞ diffeomorphisms on M satisfying properties (1), (2), (4) and also the following condition (3′): if p_1 and p_2 are periodic points such that $W^u(p_1)$ is not transversal to $W^s(p_2)$ then one of the connected components of $W^s(p_2) - \{p_2\}$ is contained in $W^u(p_1)$.

The modulus of stability of any $f \in \mathscr{m}$ is infinite. However, it may become finite if we restrict our perturbations to $\mathscr{m}$. In fact we have the following theorem.

THEOREM D. *For each integer k there is a subset $\mathscr{m}_k \subset \mathscr{m}$ such that $\mathscr{m}_k$ is open in $\mathscr{m}$ and the modulus of stability in $\mathscr{m}$ of any $f \subset \mathscr{m}_k$ is equal to k. Furthermore, $\bigcup_{k=1}^{\infty} \mathscr{m}_k$ is dense in $\mathscr{m}$.*

Using Theorem D we can describe open sets in the space of equivariant diffeomorphisms having modulus of stability equal to k for any integer k. This occurs in the case where G is a finite group and the partition of M in orbit types has at least one submanifold of dimension one.

We finish this section by stating some open questions.

Let $f \in \mathrm{Diff}^\infty(M)$ satisfy properties (1), (2), (3) and (5) above. Suppose also that if p_1 and p_2 are periodic points of f such that $W^u(0(p_1))$ is not transversal to $W^s(0(p_2))$, then there are finitely many orbits in $W^s(0(p_1))$ of transversal intersection with unstable manifolds of saddle points and there is no orbit in $W^u(0(p_2))$ belonging to stable manifold of saddle point. Is the modulus of stability of f finite?

Suppose now that there are finitely many orbits in $W^s(0(p_1))$ and in $W^u(0(p_2))$ belonging to unstable and stable manifolds of saddle points. Is the modulus of stability of f infinite? We suspect the answer is yes.

Problem. Let $f \in \mathrm{Diff}^\infty(M)$ satisfy properties (1), (2), (4) and also the following condition: there are finitely many orbits of tangency between stable and unstable manifolds of periodic points and the contact between these manifolds is of finite order. Find the modulus of stability of f.

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§2. PROOF OF THEOREMS A AND B

Let f be a C^2 diffeomorphism of a compact two dimensional manifold M having a pair of hyperbolic periodic points p_1 and p_2 such that $W^u(p_1)$ meets $W^s(p_2)$ non-transversally. Let π_i be the periods of p_i, $i = 1, 2$, and let λ_i and μ_i be the eigenvalues of $df^{\pi_i}(p_i)$ with $0 < |\lambda_i| < 1$ and $|\mu_i| > 1$. We say that $q \in W^u(p_1) \cap W^s(p_2)$ is a one side tangency point if there is a neighborhood V of q such that $W^u(p_1)$ intersects at most one connected component of $V - W^s(p_2)$. If q is a point of quasi-transversal (parabolic) intersection of $W^u(p_1)$ and $W^s(p_2)$ then q is a one side tangency point.

It follows from [4] that if there is a one side tangency point of $W^u(p_1)$ and $W^s(p_2)$ then $T_1(f, p_1, p_2) = [(\log|\mu_2|)/(\log|\lambda_1|)]$ is a conjugacy invariant. This means that if h: $M \to M$ is a conjugacy between f and $\bar{f}$, with $h(p_1) = \bar{p}_1$ and $h(p_2) = \bar{p}_2$, where $\bar{p}_i$ are hyperbolic periodic points of $\bar{f}$ such that $W^u(\bar{p}_1)$ meets $W^s(\bar{p}_2)$ non-transversally, then $T_1(f, p_1, p_2) = T_1(f, \bar{p}_1, \bar{p}_2)$. From this it follows that the modulus of stability of f is at least one. Here we will prove that the restriction of f to $W^s(p_1)$ and $W^u(p_2)$ is determined once we know the image of a single point in $W^s(p_1)$ or in $W^u(p_2)$. This is the main step in the proof of Theorem A. Furthermore we will associate to each pair of one side tangency points q_1 and q_2 a new conjugacy invariant $T_2(f, p_1, p_2, q_1, q_2)$ which will be used in the proof of Theorem B and C.

To simplify the exposition we will assume that p_1 and p_2 are fixed points and the eigenvalues λ_i and μ_i are positive. This can be done by considering, instead of f, the diffeomorphism f^n for a suitable integer n.

Let q be a one side tangency point of $W^u(p_1)$ and $W^s(p_2)$. Since f is of class C^2, there are neighborhoods U_i of p_i, with $q \in U_1$, and C^1 coordinate systems φ_i: $U_i \to R^2$ linearizing f[1]. Hence, $\varphi_i \circ f \circ \varphi_i^{-1}(x^1, x^2) = (\lambda_i x^1, \mu_i x^2)$ for $i = 1, 2$. For $p \in U_i$ we denote by (p^1, p^2) the coordinates of $\varphi_i(p)$. Let k be a positive integer such that $f^k(q) \in U_2$. Consider the mapping g defined on a neighborhood of $\varphi_1(q) = (0, q^2)$ by $g(x^1, x^2) = \varphi_2 \circ f^k \circ (\varphi_1)^{-1}(x^1, x^2)$. Let $\alpha(q, k) = [(\partial g^2)/(\partial x^1)](0, q^2)$ where g^2 is the second component of g. The following lemma is easy to prove.

Lemma 2.1. *(1) $\alpha(q, k)$ is different from zero; (2) if $q_n \to q$, $z_n \doteq f^k(q_n)$ and $[(q_n^2 - q^2)/q_n^1]$ is bounded then $[z_n^2/q_n^1]$ converges to $\alpha(q, k)$; (3) if $q_n \to q$ and $q_n^1 \neq 0$ for all n then $[(g^2(q_n^1, q_n^2) - g^2(0, q_n^2))/q_n^1]$ converges to $\alpha(q, k)$; (4) if m is a positive integer then $\alpha(q, k + m) = (\mu_2)^m \cdot \alpha(q, k)$ and $\alpha(f^{-m}(q), k + m) = (\lambda_1)^m \alpha(q, k)$.*

Lemma 2.2. *Let q_1, $q_2 \in U_1$ be one side tangency points of $W^u(p_1)$ and $W^s(p_2)$. If $\beta(f, p_1, p_2, q_1, q_2) = [\alpha(q_2, k)/\alpha(q_1, k)]$, where k is a positive integer such that $f^k(q_1)$ and $f^k(q_2)$ belong to U_2, then β doen not depend on the choices of the integer k and of the coordinate systems φ_i. Furthermore, if m_1 and m_2 are integers then*

$$\beta(f, p_1, p_2, f^{m_1}(q_1), f^{m_2}(q_2)) = \left(\frac{\mu_2}{\lambda_1}\right)^{m_2 - m_1} \beta(f, p_1, p_2, q_1, q_2).$$

Proof. Using (4) of Lemma 2.1 it is easy to see that β does not dependent on k. Let $\tilde{\varphi}_i$: $U_i \to R^2$ be others C^1 coordinate systems linearizing f. Then $\Phi_1 = \varphi_1 \circ \tilde{\varphi}_1^{-1}$ and $\Phi_2 = \tilde{\varphi}_2 \circ \varphi_2^{-1}$ are diffeomorphisms, defined on a neighborhood of the origin of R^2, which commute with the linear isomorphisms L_i: $R^2 \to R^2$, $L_i(x^1, x^2) = (\lambda_i x^1, \mu_i x^2)$, for $i = 1, 2$ respectively. Thus,

$$\frac{\partial \phi_i}{\partial x^j}(x^1, 0) = \frac{\partial \phi_i}{\partial x^j}(0, x^2) = \frac{\partial \phi_i}{\partial x^j}(0, 0)$$

for $i, j = 1, 2$ and

$$\frac{\partial \phi_i^1}{\partial x^2}(0, 0) = \frac{\partial \phi_i^2}{\partial x^1}(0, 0) = 0$$

for $i = 1, 2$. From these formulas it follows easily that

$$\alpha(f, p_1, p_2, q_i, \tilde{\varphi}_1, \tilde{\varphi}_2) = \frac{\partial \phi^2}{\partial x^2}(0, 0) \cdot \alpha(f, p_1, p_2, q_i, \varphi_i, \varphi_2) \cdot \frac{\partial \phi^1}{\partial x^1}(0, 0).$$

Therefore β does not depend on the choice of the coordinate systems. The equality in Lemma 2.2 follows easily from (4) of Lemma 2.1.

We recall that a fundamental domain for the stable manifold of a hyperbolic fixed point p of f is a compact manifold with boundary $D^s \subset W^s(p)$ such that $\bigcup_{n \in Z} f^n(D^s) = W^s(p) - \{p\}$ and, for each x in the interior of D^s, $f^n(x)$ does not belong to

D^s if $n \neq 0$. If p is a saddle point in dimension 2 then D^s is either a closed interval or a pair of closed intervals.

LEMMA 2.3. *Let $f \in \mathrm{Diff}^2(M)$ be as above. Suppose that $[(\log \mu_2)/(\log \lambda_1)]$ is irrational. Let $q \in U_1$ be a one side tangency point of $W^u(p_1)$ and $W^s(p_2)$. Let $w \in W^u(p_2)$. Then there exist a set G_w, contained in a fundamental domain D^s of $W^s(p_1)$, with the following properties: (a) G_w is dense in one of the connected components of D^s; (b) for each $x \in G_w$ there are sequences $x_j \to x$, $m_j \to \infty$ and $n_j \to \infty$ such that $q_j = f^{m_j}(x_j)$ converges to q, $[(q_j^2 - q^2)/q_j^1]$ is bounded and $f^{n_j+k}(q_j)$ converges to w where k is an integer such that $f^k(q_j) \in U_2$.*

Proof. Let $z = f^k(q)$ and $S \subset U_2$ be a 1-dimensional disc transversal to $W^s(p_2)$ at z. It is clear that there exist a sequence z_n in S such that $f^n(z_n)$ converges to $w = (0, w^2)$. Hence $z_n^2 = (\mu_2)^{-n} w^2$. If $q_n = f^{-k}(z_n)$ then q_n converges to q and $z_n^2 = \alpha_n q_n^1$. Since $q_n \in f^{-k}(S)$ which is transversal to $W^u(p_1)$ it follows that $[(q_n^2 - q^2)/q_n^1]$ converges. In particular $[(q_n^2 - q^2)/q_n^1]$ is bounded and α_n converges to $\alpha = \alpha(q, k)$. We may assume that z_n^2 and q_n^1 are positive and $\alpha(q, k) > 0$. It suffices to prove that the set of limit points of the double sequence $f^{-j}(q_n)$ is dense in $D_+^s = \{(x^1, 0) \in U_1;\ \lambda_1 \leq x^1 \leq 1\}$. Let $s_n = [(\log q_n^1)/(\log \lambda_1)]$. Then

$$s_n = \frac{\log \frac{1}{\alpha} w^2}{\log \lambda_1} + \frac{\log \frac{\alpha}{\alpha_n}}{\log \lambda_1} - n \frac{\log \mu_2}{\log \lambda_1}.$$

Hence

$$s_n = a + b_n - n \frac{\log \mu_2}{\log \lambda_1}$$

with b_n converging to zero. For each n we may write $s_n = \tilde{s}_n + m_n$ where $\tilde{s}_n \in [0, 1]$ and m_n is an integer. Hence $m_n \to \infty$ and since $[(\log \mu_2)/(\log \lambda_1)]$ is irrational, it follows that $\{\tilde{s}_n, n \geq 0\}$ is dense in $[0, 1]$. Hence the set $\tilde{G}$ of limit points of the sequence $\tilde{s}_n$ is dense in $[0, 1]$. Consequently the set $G_w = \{(x^1, 0);\ x^1 = (\lambda_1)^\delta$ for some $\delta \in \tilde{G}\}$ is dense in D_+^s. We claim that any point in G_w is a limit point of the double sequence $f^{-j}(q_n)$. In fact, if $x \in G_w$ then $x = (x^1, 0)$ with $x^1 = (\lambda_1)^\delta$ and $\delta = \lim_{j\to\infty} \tilde{s}_{n_j}$. Hence

$$x^1 = \lim_{j\to\infty} (\lambda_1)^{\tilde{s}_{n_j}} = \lim_{j\to\infty} (\lambda_1)^{s_{n_j}} (\lambda_1)^{-m_{n_j}} = \lim_{j\to\infty} (\lambda_1)^{-m_{n_j}}$$
$$\times \exp\left(\frac{\log q_{n_j}^1}{\log \lambda_1} \cdot \log \lambda_1\right) = \lim_{j\to\infty} (\lambda_1)^{-m_{n_j}} q_{n_j}^1.$$

Therefore $x_{n_j} = f^{-m_{n_j}}(q_{n_j})$ converges to x as $j \to \infty$. This proves the claim and the lemma.

Remarks. If μ_2 is positive, λ_1 negative and $[(\log \mu_2)/(\log |\lambda_1|)]$ is irrational then G_w is dense in the fundamental domain D^s. If both eigenvalues are negative and $[(\log |\mu_2|)/(\log |\lambda_1|)]$ is irrational the lemma remains true.

Let $\bar{f}$ be another C^2 diffeomorphism having a one side tangency point $\bar{q}$ of $W^u(\bar{p}_1)$ and $W^s(\bar{p}_2)$. Let $0 < \bar{\lambda}_i < 1$ and $\bar{\mu}_i > 1$ be the eigenvalues of $d\bar{f}(\bar{p}_i)$, $i = 1, 2$. As before we consider C^1 coordinate systems in neighborhoods $\bar{U}_i$ of $\bar{p}_i$, $\bar{U}_1 \supset \bar{q}$, linearizing $\bar{f}$. We denote by $(\bar{x}^1, \bar{x}^2)$ the coordinates of a point $x \in \bar{U}_i$.

LEMMA 2.4. *Let $h: M \to M$ be a conjugacy between f and $\bar{f}$ such that $h(p_i) = \bar{p}_i$ and $h(q) = \bar{q}$. Then there exist constants a_-, a_+, b_-, b_+ such that:*

$$h(x^1, 0) = (a_+(x^1)^c, 0) \quad \text{if} \quad (x^1, 0) \in U_1 \cap W^s(p_1) \quad \text{and} \quad x^1 \geq 0;$$

$$h(x^1, 0) = (a_-|x^1|^c, 0) \quad \text{if} \quad (x^1, 0) \in U_1 \cap W^s(p_1) \quad \text{and} \quad x^1 \leq 0;$$

$$h(0, y^2) = (0, b_+(y^2)^c) \quad \text{if} \quad y \in U_2 \cap W^u(p_2) \quad \text{and} \quad y^2 \geq 0;$$

$$h(0, y^2) = (0, b_-|y^2|^c) \quad \text{if} \quad y \in U_2 \cap W^u(p_2) \quad \text{and} \quad y^2 \leq 0.$$

Here

$$c = \frac{\log \bar{\mu}_2}{\log \mu_2}.$$

Proof. Let $V_q \in U_1$ and $V_z \subset U_2$ be small neighborhoods of q and $z = f^k(q)$ respectively. We may choose the coordinates in U_1 and U_2 in such a way that $V_q \cap W^s(p_2) \subset \{x \in V_q; x^1 \geq 0\}$ and $V_z \cap W^u(p_1) \subset \{x \in V_z; x^2 \leq 0\}$. Similarly we can choose the coordinates in $\bar{U}_i$ in such a way that $V_{\bar{q}} \cap W^s(\bar{p}_2) \subset \{x \in V_{\bar{q}}; x^1 \geq 0\}$ and $V_{\bar{z}} \cap W^u(\bar{p}_1) = \{x \in V_{\bar{z}}; x^2 \leq 0\}$, where $V_{\bar{q}}$ and $V_{\bar{z}}$ are small neighborhoods of $\bar{q}$ and $\bar{z}$ respectively. Let h be a conjugacy between f and $\bar{f}$ sending p_i on $\bar{p}_i$ and q on $\bar{q}$. For $p \in M$ we denote by $\bar{p}$ the image of p under h. Since $h(W^s(p_i)) = W^s(\bar{p}_i)$, $h(W^u(p_i)) = W^u(\bar{p}_i)$ it follows that if p is near q and p^1 is positive then $\bar{p}^1$ is positive; if p is near z and p^2 is positive then $\bar{p}^2$ is positive. Let w be a point in $W^u(p_2)$ with $w^2 > 0$. Let G_w be as in Lemma 2.3. Let $x \in G_w$, $x_j \to x$, $m_j \to \infty$ and $n_j \to \infty$ be such that $q_j = f^{m_j}(x_j)$ converges to q and $w_j = f^{k+n_j}(q_j) = f^{n_j}(z_j)$ converges to w. Hence q_j^1 and z_j^2 are positive and $z_j^2 = \alpha_j q_j^1$ where α_j converges to $\alpha = \alpha(q, k)$. Since $w_j^2 = \alpha_j \mu_2^{n_j} \lambda_1^{m_j} x_j^1$ converges to w^2 and x_j^1 converges to x^1, it follows that $\mu_2^{n_j}\lambda_1^{m_j}$ converges to $(w^2/\alpha x^1)$. Using the fact that h is a conjugacy we conclude that $\bar{x}_j$ converges to $\bar{x} = (\bar{x}^1, 0)$, $\bar{q}_j = \bar{f}^{m_j}(\bar{x}_j)$ converges to $\bar{q}$, $\bar{z}_j = \bar{f}^k(\bar{q}_j)$ converges to $\bar{z}$ and $\bar{w}_j = \bar{f}^{n_j}(\bar{z}_j)$ converges to $\bar{w}$. Furthermore $\bar{z}_j^2$ and $\bar{q}_j^1$ are positive. Hence $\bar{w}_j^2 = d_j \bar{\mu}_2^{n_j} \bar{\lambda}_1^{m_j} \bar{x}_j^1$ where $d_j = (\bar{z}_j^2 / \bar{q}_j^1)$ is positive. Since $[(\log \bar{\mu}_2)/(\log \mu_2)] = [(\log \bar{\lambda}_1)/(\log \lambda_1)] = c$ it follows $\bar{\mu}_2^{n_j}\bar{\lambda}_1^{m_j} = (\mu_2^{n_j}\lambda_1^{m_j})^c$ converges to $(w^2/\alpha x^1)^c$. Thus d_j converges to $(\bar{w}^2/\bar{x}_1) \cdot [\alpha^c (x^1)^c / (w^2)^c]$. We claim that $\lim_{j \to \infty} d_j$ is less or equal $\bar{\alpha}(\bar{f}, \bar{p}_1, \bar{p}_2, \bar{q}, k)$. In fact, if $\bar{g}^2$ is the second component of the mapping $\bar{g} = \bar{\varphi}_2 \circ \bar{f}^k \circ \bar{\varphi}_1^{-1}$ then

$$\frac{\bar{z}_j^2}{\bar{q}_j^1} = \frac{\bar{g}^2(\bar{q}_j^1, \bar{q}_j^2)}{\bar{q}_j^1} \leq \frac{\bar{g}^2(\bar{q}_j^1, \bar{q}_j^2) - \bar{g}^2(0, \bar{q}_j^2)}{\bar{q}_j^1}$$

because $g^2(0, q_j^2)$ is negative. By Lemma 2.1, the right member converges to $\bar{\alpha}$. Hence

$$\lim_{j \to \infty} d_j = \frac{\bar{w}^2}{\bar{x}^1} \frac{\alpha^c (x^1)^c}{(w^2)^c} \leq \bar{\alpha}.$$

Therefore $\bar{x}^1 \geq a_+(x^1)^c$ for every $x \in G_w$ where $a_+ = [\bar{w}^2 \cdot \alpha^c/(w^2)^c \cdot \bar{\alpha}]$. Since G_w is dense in one of the connected components of a fundamental domain, the inequality remains true for every $x \in W^s(p_1)$ with $x^1 \geq 0$. By reasoning with h^{-1} instead of h we prove that $(x^1)^c \geq (1/a_+)\bar{x}_1$ for every $\bar{x} \in W^s(\bar{p}_1)$ with $\bar{x}^1 \geq 0$. Therefore $\bar{x}^1 = a_+(x^1)^c$ for every $x \in W^s(p_1)$ with $x^1 \geq 0$. Now, if $y \in W^u(p_2)$ is such that $y^2 > 0$ then we conclude, using the same arguments as above, that $a_+ = [\bar{y}^2 \cdot \alpha^c/(y^2)^c\bar{\alpha}]$. Hence $\bar{y}^2 = b_+(y^2)^c$ where $b_+ = [a_+\bar{\alpha}/(\alpha)^c]$. To prove the existence of the constants a_- and b_- we use the same reasoning with f^{-1} and $\bar{f}^{-1}$. Thus the lemma is proved.

LEMMA 2.5. *Let q_1, q_2 be one side tangency points of $W^s(p_2)$ and $W^u(p_1)$. Let h: $M \to M$ be a conjugacy between f and $\bar{f}$ such that $h(p_i) = \bar{p}_i$ and $h(q_i) = \bar{q}_i$, $i = 1, 2$, where q_i are one side tangency points of $W^s(\bar{p}_2)$ and $W^u(\bar{p}_1)$. If $[(\log \mu_2)/(\log \lambda_1)]$ is irrational them $|\beta(\bar{f}, \bar{p}_1, \bar{p}_2, \bar{q}_1, \bar{q}_2)| = |\beta(f, p_1, p_2, q_1, q_2)|^c$ where $c = [(\log \bar{\mu}_2)/(\log \mu_2)]$ and β is as in Lemma 2.2.*

Proof. Let $w \in W^u(p_2)$ and $x \in W^s(p_1)$ be such that $w^2 > 0$ and $x^1 > 0$. Since q^1 is a one side tangency point, it follows from the proof of Lemma 2.4 that

$$\frac{\bar{x}^1}{(x^1)^c} = a_+ = \frac{\bar{w}^2 \cdot |\alpha(q_1, k)|^c}{(w^2)^c|\bar{\alpha}(\bar{q}_1, k)|}.$$

On the other hand, q_2 is also a one side tangency point and, therefore,

$$a_+ = \frac{\bar{w}^2|\alpha(q_2, k)|^c}{(w^2)^c|\bar{\alpha}(\bar{q}_2, k)|}.$$

Thus

$$\begin{aligned} |\beta(\bar{f}, \bar{p}_1, \bar{p}_2, \bar{q}_1, \bar{q}_2)| &= \frac{|\bar{\alpha}(\bar{q}_2, k)|}{|\bar{\alpha}(\bar{q}_1, k)|} = \left|\frac{\alpha(q_2, k)}{\alpha(q_1, k)}\right|^c \\ &= |\beta(f, p_1, p_2, q_1, q_2)|^c. \end{aligned}$$

Since the sign of β is also preserved by a conjugacy we have the following corollary.

COROLLARY. *Let q_1, q_2 be one side tangency points of $W^u(p_1)$ and $W^s(p_2)$. Let $T_2(f, p_1, p_2, q_1, q_2)$ be equal to $\pm|\beta(f, p_1, p_2, q_1, q_2,)|^{1/(\log|\lambda_1|)}$ and having the same sign as β. If $[(\log|\mu_2|)/(\log|\lambda_1|)]$ is irrational then T_2 is a conjugacy invariant.*

Proof of Theorem B. Let f be a C^2 diffeomorphism having two hyperbolic periodic points p_1, p_2 such that $W^u(p_1) \cap W^s(p_2)$ contains k orbits of tangency. We can approximate f by a C^2 diffeomorphism $\bar{f}$ which has two hyperbolic periodic points $\bar{p}_1$, $\bar{p}_2$, near p_1 and p_2 resp., such that $W^u(\bar{p}_1) \cap W^s(\bar{p}_2)$ contains k orbits of quasi-transversal intersection and $[(\log|\bar{\mu}_2|)/(\log|\bar{\lambda}_1|)]$ is irrational, where $\bar{\lambda}_i$ and $\bar{\mu}_i$ are the eigenvalues of the periodic points $\bar{p}_i$ with $|\bar{\lambda}_i| < 1$ and $|\bar{\mu}_i| > 1$, $i = 1, 2$. Since a point of quasi-transversal intersection is a one side tangency point, it follows from Lemma 2.5 that the modulus of stability of $\bar{f}$ is at least k. Hence the modulus of stability of f is at least k and the theorem is proved.

Proof of Theorem A. Let f be a C^2 diffeomorphism having two periodic points p_1, p_2 such that $W^u(p_1) \cap W^s(p_2)$ contains an orbit of quasi-transversal intersection. Suppose that $[(\log|\mu_2|)/(\log|\lambda_1|)]$ is irrational where λ_i and μ_i are the eigenvalues of $df^{n_i}(p_i)$, $i = 1, 2$, with $|\lambda_i| < 1$ and $|\mu_i| > 1$. If $W^s(p_1)$ contains k orbits whose α-limit are hyperbolic periodic points of f of saddle type then it follows easily from Lemma 2.4 that the modulus of stability of f is at least k. It is easy to see that the same holds if $k = \infty$.

§3. PROOF OF THEOREM C

Let U_k be the set of C^∞ diffeomorphisms satisfying properties (1)–(4) of §1. Let us prove that the modulus of stability of $f \in U_k$ is equal to k. To simplify the exposition we assume that the k orbits of tangency are in $W^u(p_1) \cap W^s(p_2)$ where p_1 and p_2 are fixed points of f. We assume also that the eigenvalues λ_i and μ_i of $df(p_i)$ are positive, $\lambda_i < 1$ and $\mu_i > 1$. The proof when there are orbits of tangency between stable and unstable manifolds of several pairs of periodic orbits is similar.

Let $D^s \subset W^s(p_2)$ be a fundamental domain such that no unstable manifold of saddle point of f meets the boundary of D^s. Let $z_1, \dots, z_k \in D^s$ be the tangency points of $W^u(p_1) \cap W^s(p_2)$ in D^s. Let $V_1, \dots, V_k$ be disjoint neighborhoods of $z_1, \dots, z_k$ such that z_i is the only point in V_i belonging to unstable manifold of saddle point of f.

If $\mathscr{n}$ is a small neighborhood of f then each $g \in \mathscr{n}$ has fixed points $p_i(g)$ near $p_i = p_i(f)$ and the eigenvalues $\lambda_i(g)$ and $\mu_i(g)$ of $dg(p_i(g))$ are near λ_i and μ_i respectively. Furthermore, for $i = 1, \dots, k$, $W^u(p_1(g)) \cap W^s(p_2(g)) \cap V_i$ is either empty or a pair of points of transversal intersection or a point $z_i(g)$ of quasi-transversal intersection. It is easy to see that if $\mathscr{n}$ is small enough then any $g \in \mathscr{n}$ is either a Morse–Smale or an element of U_j for some $j \leq k$.

For each $g \in \mathscr{n}$ we choose a fundamental domain D_g^s of $W^s(p_2(g))$ such that D_g^s varies continuously with g. For any sequence of integers $I = (I_1, \dots, I_j)$ with $1 \leq I_1 < I_2 < \cdots < I_j \leq k$ we denote by $\mathscr{n}_I$ the set of diffeomorphisms $g \in \mathscr{n}$ such that the points of tangency of $W^u(p_1(g))$ and $W^s(p_2(g))$ in D_g^s are $z_{I_1}(g), \dots, z_{I_j}(g)$. Let $T_1(g)$ denote the number $[(\log \mu_2(g))/(\log \lambda_1(g))]$ and let $T_2(z_{I_1}(g), z_{I_j}(g)), \dots, T_2(z_{I_{j-1}}(g), z_{I_j}(g))$ be the conjugacy invariants defined in §2.

It is easy to see that Theorem C follows from the theorem below.

THEOREM 3.1. *Let $g, \bar{g} \in \mathscr{n}_I$ be such that for each $i = 1, \dots, j$, the number of points in $V_i \cap W^s(p_2(g)) \cap W^u(p_1(g))$ is equal to the number of points in $V_i \cap W^s(p_2(\bar{g})) \cap W^u(p_1(\bar{g}))$. If $T_1(g) = T_1(\bar{g})$ and $T_2(z_{I_m}(g), z_{I_j}(g)) = T_2(z_{I_m}(\bar{g}), z_{I_j}(\bar{g}))$ for $m = 1, \dots, j-1$ then g is topologically equivalent to $\bar{g}$.*

To prove Theorem 3.1 we need several geometric constructions.

Recall that an unstable tubular family for a periodic point p of g is a continuous retraction π^u: $V^u \to W^s(0(p))$, where V^u is a neighborhood of $W^u(0(p))$, such that $(\pi^u)^{-1}(p) = W^u(p)$; $\pi^u(g(x)) = g\pi^u(x)$ whenever x and $g(x) \in V^u$; the fibers of π^u are C^1 submanifolds transversal to $W^s(0(p))$ and the mapping which assigns to each $x \in V^u$ the tangent space of the fiber through x is continuous. Similarly we define a stable tubular family π^s: $V^s \to W^u(0(p))$. Let $p_1, p_2, \dots, p_s$ be the periodic points of g, of saddle type, with $p_1 = p_1(g)$ and $p_2 = p_2(g)$. For each $l > 2$ we construct a tubular family π_l^u: $V_l^u \to W^s(0(p_l))$ satisfying the following compatibility condition: if $W^u(0(p_l))$ intersects $W^s(0(p_i))$ then any fiber of π_i^u which intersects a fiber of π_l^u is contained in this fiber. This can be done by induction in the phase diagram of g as in [5] or [6]. For p_1 and p_2 we will consider tubular families π_i^u: $V_i^u \to W^s(p_i)$ and π_i^s: $V_i^s \to W^u(p_i)$, $i = 1, 2$, which are of class C^2. For this we take C^2 coordinate systems in a neighborhood U_i of p_i linearizing g. If $x \in U_i$ we denote by (x^1, x^2) the coordinates of x. Hence $g(x) = (\lambda_i(g)x^1, \mu_i(g)x^2)$. In these coordinates we define $\pi_i^u(x) = x^1$ and $\pi_i^s(x) = x^2$. It is easy to extend π_i^u to a neighborhood V_i^u of $W^u(p_i)$.

We may assume that $D_g^s \subset U_2$. Since for each $m = 1, \dots, j$, $z_{I_m}(g)$ is a quasi-transversal point of intersection of $W^u(p_1)$ and $W^s(p_2)$ and the tubular families π_1^u

and $\pi_2{}^s$ are C^2, it follows that, in a neighborhood of $z_{I_m}(g)$, the fibers of $\pi_1{}^u$ are transversal to the fibers of $\pi_2{}^s$ except along a C^1 curve Σ_m which is transversal to $W^s(p_2)$. *Let* $J_m \subset D_g{}^s$ be an interval containing z_{I_m}, so small that z_{I_m} is the only point in J_m whose α-limit set is a saddle point. Shrinking $V_1{}^u$ if necessary we can assume that $V_1{}^u \cap D_g{}^s$ is contained in $\bigcup_{m=1}^{j} J_m$. Through the points of the boundary of J_m we take a pair of discs $S_m{}^1$, $S_m{}^2$ contained in the fibers of $\pi_1{}^u$. In the strip bounded by $S_m{}^1$ and $S_m{}^2$ we consider a foliation, having these discs as leaves, satisfying the following conditions: all leaves are transversal to the fibers of $\pi_2{}^s$ except along Σ_m and the foliation coincides with the fibration $\pi_1{}^u$ in $V_1{}^u$. Over $D_g{}^s - \bigcup_{m=1}^{j} J_m$ we raise a continuous fibration, whose fibers are discs transversal to the fibers of $\pi_2{}^s$, satisfying the properties: the fibers over the points in the boundary of $D_g{}^s$ is contained in the fibers of $\pi_2{}^u$; the fiber over a point which belongs to $V_i{}^u$, for some $i \neq 2$, is contained in the fiber of $\pi_i{}^u$ through this point. Now we iterate this foliation by g. Let us denote by $\mathcal{F}^u$ the foliation constructed above. We notice that the leaves of $\mathcal{F}^u$ in the strips bounded by $S_m{}^2$ and S^1_{m+1} are discs transversal to $W^s(p_2)$.

We perform all the constructions above for $\bar{g}$. Hence $\bar{\Sigma}_m$ are the curves of tangency of the unstable tubular family of $\bar{p}_1 = p_1(\bar{g})$ with the stable tubular family of $\bar{p}_2 = p_2(\bar{g})$, $\bar{z}_{I_m} = z_{I_m}(\bar{g}) = \bar{\Sigma}_m \cap D_{\bar{g}}{}^s$ and $\bar{\mathcal{F}}^u$ denote the foliation whose leaf through any point in $\bar{V}_i{}^u$, $i \neq 2$, is contained in the fiber of $\bar{\pi}_i{}^u$ through this point.

Let us define the conjugacy h between g and $\bar{g}$. For each $i > 2$ we construct a homeomorphism h_i: $W^s(0(p_i)) \to W^s(0(\bar{p}_i))$, where $\bar{p}_i$ is the periodic point of $\bar{g}$ near p_i, such that $h_i g = \bar{g} h_i$ and h_i preserves the unstable tubular families of the periodic points. This means that if $x \in W^s(p_i) \cap V_n{}^u$ for $i, n > 2$ then $\bar{\pi}_n{}^u(h_i(x)) = h_n(\pi_n{}^u(x))$. This can be done by induction on the phase diagrams of g and $\bar{g}$ as in the proof of the stability of Morse–Smale diffeomorphisms[5]. Now we define a homeomorphism h_1: $W^s(p_1) \to W^s(\bar{p}_1)$ conjugating g and $\bar{g}$. Let $x \in W^s(p_1)$ be in a small neighborhood of p_1 contained in U_1. Then $x = (x^1, 0)$ in the coordinate system in U_1 which linearizes g. We set $h_1(x) = \bar{x}$ whose coordinates $(\bar{x}^1, 0)$, in the coordinate system linearizing $\bar{g}$, are given by $\bar{x}^1 = (x^1)^c$ if $x^1 \geq 0$ and $\bar{x}^1 = -|x^1|^c$ if $x^1 < 0$, where $c = [(\log \bar{\lambda}_1)/(\log \lambda_1)]$. Clearly h_1 is a homeomorphism of a neighborhood N of p_1 in $W^s(p_1)$ onto a neighborhood of $\bar{p}_1$ in $W^s(\bar{p}_1)$ and $h_1 g = \bar{g}^{h_1}$. Now we can extend h_1 to $W^s(p_1)$: if $x \in W^s(p_1)$ then there exists an integer n such that $g^n(x) \in N$; we then set $h_1(x) = \bar{g}^{-n} h g^n(x)$

For each $m = 1, \ldots, j$, we denote by φ_m: $\Sigma_m \to W^s(p_1)$ and by $\bar{\varphi}_m$: $\bar{\Sigma}_m \to W^s(\bar{p}_1)$ the restrictions of $\pi_1{}^u$ and $\bar{\pi}_1{}^u$ respectively. Then φ_m and $\bar{\varphi}_m$ are C^1 diffeomorphisms onto neighborhoods of p_1 and $\bar{p}_1$ respectively. We define homoemorphisms h_{Σ_m}: $\Sigma_m \to \bar{\Sigma}_m$ by $\bar{z} = h_{\Sigma_m}(z) = \bar{\varphi}_m{}^{-1} h_1 \varphi_m(z)$. Now we extend h_{Σ_m} to $\bigcup_{n \in Z} g^n(\Sigma_m)$ using the equation $h_{\Sigma_m} g = \bar{g} h_{\Sigma_m}$. We claim that there is a homeomorphism h^u: $W^u(p_2) \to W^u(\bar{p}_2)$ such that: (i) $h^u g = \bar{g} h^u$; (ii) if $w_i \to w \in W^u(p_2)$, $w_i \in g^{n_i}(\Sigma_m)$ when $w_i = h_{\Sigma_m}^{(w_i)}$ converges to $\bar{w} = h^u(\omega)$, for all $m = 1, \ldots, j$. In fact, let $w = (0, w^2)$ and $l > 0$ be an integer such that $g^{-1}(\Sigma_m) \subset U_1$ and $\bar{g}^{-l}(\bar{\Sigma}_m) \subset U_1$ for $m = 1, \ldots, j$. If $y_i = g^{-(l+n_i)}(w_i)$ and $\bar{y}_i = h_{\Sigma_m}(y_i) = \bar{g}^{-(l+n_i)}(w_i)$ then $\bar{y}_i{}^1 = \pm |y_i{}^1|^c$. Hence $\bar{w}_i$ converges to $\bar{w} = (0, \bar{w}^2)$ where $\bar{w}^2 = \pm b |w^2|^c$ and $b = [|\alpha(\bar{g}^{-1}(\bar{z}_{I_m}), l| / |\alpha(g^{-1}(z_{I_m}), l|^c]$. Since $T_2(z_{I_m}(g), z_{I_j}(g)) = T_2(z_{I_m}(\bar{g})$ for $m = 1, \ldots, j-1$, it follows that b does not depend on m. Hence we may define $h^u(w) = \bar{w}$ by $\bar{w}^2 = b|w^2|^c$ if $w^2 \geq 0$ and $\bar{w}^2 = -b|w^2|^c$ if $w^2 \leq 0$. Since $[(\log \bar{\mu}_2)/(\log \mu_2)] = c$, h^u is a homeomorphism of a neighborhood of p_2 in $W^u(p_2)$ onto a neighborhood of $\bar{p}_2$ in $W^u(\bar{p}_2)$, conjugating g and $\bar{g}$. Now we can extend h^u to $W^u(p_2)$ using the equation $h^u g = \bar{g} h^u$. The claim is proved.

Next we define a homeomorphism h_2: $W^s(p_2) \to W^s(\bar{p}_2)$ such that $h_2 g = \bar{g} h_2$ and h_2 preserves the unstable tubular families π_i^u and $\bar{\pi}_i^u$ for $i \neq 2$. Let $A_m \subset D_g^s$ be a neighborhood of z_{l_m} such that for any $z \in A_m$ the leaf of π_1^u through z intersects Σ_m in a point $\gamma_m(z)$. Let $A_m^+ \subset A_m$ be the set of points $z \in A_m$ such that $z^1 > z^1_{l_m}$ where $(z^1, 0)$ and $(z^1_{l_m}, 0)$ are the coordinates of z and z_{l_m} respectively. The restriction γ_m^+ of γ_m to A_m^+ is a homeomorphism onto an open subset of Σ_m. Similarly we consider a neighborhood $\bar{A}_m$ of $\bar{z}_{l_m}$ in $W^s(\bar{p}_2)$ and a mapping $\bar{\gamma}_m$: $\bar{A}_m \to \bar{\Sigma}_m$. For $z \in A_m^+$ we set $h_2(z) = (\bar{\gamma}_m^+)^{-1} h_{\Sigma_m} \gamma_m(z)$. If $z \in A_m - A_m^+$ we define $h_2(z) = (\bar{\gamma}_m^-)^{-1} h_{\Sigma_m} \gamma_m(z)$, where $\bar{\gamma}_m^-$ is the restriction of $\bar{\gamma}_m$ to $\bar{A}_m - \bar{A}_m^+$. Hence h_2: $\bigcup_{m=1}^{j} A_m \to \bigcup_{m=1}^{j} \bar{A}_m$ is a homeomorphism and preserves the unstable tubular families π_1^u and $\bar{\pi}_1^u$. Let p_i be a periodic point of g with behavior 1 with respect to p_2 (see [5], p. 389). Then $W^u(p_i)$ intersects D_g^s in finitely many points $q_1, \ldots, q_\delta$ which are contained in $D_g^s - \bigcup_{m=1}^{j} A_m$. The restriction of π_i^u to neighborhoods $B_{i,l} \subset D_g^s$ of q_l are homeomorphisms $\varphi_{i,l}$ of $B_{i,l}$ onto a neighborhood of p_i in $W^s(p_i)$. For $z \in B_{i,l}$ we define $h_2(z) = (\bar{\varphi}_{i,l})^{-1} h_i \varphi_{i,l}(z)$ where $\bar{\varphi}_{i,l}$ is the restriction of $\bar{\pi}_i^u$ to a neighborhood $\bar{B}_{i,l}$ of $\bar{q}_l$ in $W^s(\bar{p}_2)$. Here $\bar{q}_1, \ldots, \bar{q}_\delta$ are the intersection of $W^u(\bar{p}_i)$ with $W^s(\bar{p}_2)$, $\bar{p}_i$ is the periodic point of $\bar{g}$ near p_i and $\bar{q}_l$ is near q_l for $l = 1, \ldots, \delta$, since $\bar{g}$ is near g. Repeating the construction for every periodic point of g whose behavior with respect to p_2 is 1 we extend h_2 to a homeomorphism h_2: $A^1 \to \bar{A}^1$ preserving the unstable tubular families, where $A^1 \subset D_g^s$ is an open set which contains the intersection with D_g^s of all the unstable manifolds of periodic points with behavior one with respect to p_2. If p is a periodic point of g of behavior 2 with respect to p_2 then $W^u(p)$ intersects $D_g^s - A^1$ in finitely many points. So we can, using the same arguments, extend h_2 to an open set A^2 which contains A^1 and the intersection of D_g^s with the unstable manifolds of all periodic points with behavior 2 with respect to p_2. By induction, we extend h_2 to an open set A in D_g^s which contains the intersections of D_g^s with the unstable manifolds of all saddle points of g, in such way that h_2 preserves the unstable tubular families π_i^u and $\bar{\pi}_i^u$ for every $i \neq 2$. It is clear from the construction above that A is the union of finitely many intervals. Furthermore, since g and $\bar{g}$ are close to each other and we can construct tubular families depending continuously on the diffeomorphism, it follows that $h_2(x)$ is close to x for every $x \in A$. Hence we can extend h_2 to a homeomorphism h_2: $D_g^s \to D_{\bar{g}}^s$. Next we extend h_2 to homeomorphism of $W^s(p_2)$ onto $W^s(\bar{p}_2)$ using the equation $h_2 g = \bar{g} h_2$.

Up to now we have defined a homeomorphism h on the union of the stable manifolds of the saddle points of g, conjugating g and $\bar{g}$ and preserving the tubular families π_i^u and $\bar{\pi}_i^u$ for $i \neq 2$. This homeomorphism is also defined on $\bigcup_{m=1}^{j} \bigcup_{n \in Z} g^n(\Sigma_m)$, on $W^u(p_2)$ and preserves the foliations $\mathcal{F}^u$ and $\bar{\mathcal{F}}^u$. To extend h to a neighborhood of p_2 we are going to construct a continuous retraction π: $\bar{N}_2^u \to W^u(\bar{p}_2)$, where $\bar{N}_2^u$ is a neighborhood of $W^u(\bar{p}_2)$ containing $D_{\bar{g}}^s$, having the following properties:

(i) $\pi^{-1}(\bar{p}_2) = W^s(\bar{p}_2) \cap \bar{N}_2^u$;

(ii) π is $\bar{g}$ invariant, namely, $\pi(\bar{g}(p)) = \bar{g}(\pi(p))$;

(iii) If $q \in \Sigma_m$ then $\pi(h(q)) = h(\pi_2^s(q))$ for $m = 1, \ldots, j$;

(iv) The fibers of π in $U_2 \cap \bar{N}_2^u$ coincide with the fibers of $\bar{\pi}_2^s$ except on the iterates of the strip bounded by $\bar{S}_m^2$ and $\bar{S}^1_{m+1}$, $m = 1, \ldots, j-1$.

(v) In the interior of the strips bounded by $\bar{S}_m^2$ and $\bar{S}^1_{m+1}$ the fibers of π are differentiable and transversal to the leaves of the foliation $\bar{\mathcal{F}}^u$.

Before proving the existence of π we extend h to M.

It is easy to see that there is a unique extension of $h_2 = h|W^s(p_2)$ to a neighborhood N^s of D_g^s satisfying the following conditions: (a) $\pi(h(x)) = h(\pi_2^s(x))$; (b) $x, y \in N^s$ belong to the same leaf of $\mathcal{F}^u$ iff $h(x)$ and $h(y)$ belong to the same leaf of $\bar{\mathcal{F}}^u$; (c) if x and $g(x)$ are in N^s then $\bar{g}h(x) = hg(x)$. Now let $N_2^u = \bigcup_{n\geq 0} g^n(N^s) \cup W^u(p_2)$. Then N_2^u is a neighborhood of $W^u(p_2)$[5], and we can extend h to N_2^u by $h(x) = h^u(x)$ if $x \in W^u(p_2)$ and $h(x) = \bar{g}^n hg^{-n}(x)$ if $g^{-n}(x) \in N^s$. We claim that h is a homeomorphism of N_2^u onto a neighborhood $\bar{N}_2^u$ of $W^u(\bar{p}_2)$. Clearly, the restriction of h to $N_2^u - W^u(p_2)$ is a homeomorphism onto $\bar{N}_2^u - W^u(\bar{p}_2)$. So it remains to prove the continuity of h at $W^u(p_2)$. Let $x_i \in N_2^u - W^u(p_2)$ be a sequence converging to $x \in W^u(p_2)$. Then there exists a sequence $n_i \to \infty$ such that $g^{-n_i}(x_i) \in N^s$. Since $\pi_2^s(x_i) \to x$ and $h|W^u(p_2) = h^u$ is continuous it follows that $\pi h(x_i) = h\pi_2^s(x_i) = h^u(\pi_2^s(x_i))$ converges to $h(x)$. We have also that $\bar{\pi}_2^u(h(x_i))$ converges to zero because $n_i \to \infty$. Hence $h(x_i)$ converges to $h(x)$. Thus h is continuous. It is easy to see that h has a continuous inverse. Therefore h: $N_2^u \to \bar{N}_2^u$ is a homeomorphism. We notice that if $x \in N_2^u \cap V_i^u$ for some $i \neq 2$ then $\bar{\pi}_i^u h(x) = h_i \pi_i^u(x)$.

To extend h to M we proceed as in the proof of the stability of Morse–Smale diffeomorphisms. Let $Q_1, \dots, Q_t$ be the sinks of g and let $\bar{Q}_1, \dots, \bar{Q}_t$ be the corresponding sinks of $\bar{g}$. For each $i = 1, \dots, t$ we choose a fundamental domain $G_g^s(Q_i)$ of Q_i whose boundary is transversal to the unstable manifolds of the saddle points of g. We also choose a fundamental domain $G_{\bar{g}}^s(\bar{Q}_i)$ near $G_g^s(Q_i)$ such that $G_{\bar{g}}^s(\bar{Q}_i)$ contains $h(N_2^u \cap G_g^s(Q_i))$ whenever $G_{\bar{g}}^s(\bar{Q}_i)$ intersects $\bar{N}_2^u$. Using the methods of [5] we extend h to a homeomorphism h: $G_g^s(Q_i) \to G_{\bar{g}}^s(\bar{Q}_i)$, for $i = 1, \dots, t$, such that: (i) if $x, g(x) \in G_g^s(Q_i)$ then $hg(x) = \bar{g}h(x)$; (ii) if $x \in V_n^u \cap G_g^s(Q_i)$ for some n then $\bar{\pi}_n^u(h(x)) = h_n(\pi_n^u(x))$. Now we can extend h to M. If $x \in W^s(p_l)$ for some l we define $h(x) = h_l(x)$; if $g^n(x) \in G_g^s(Q_i)$ for some n and some i then we set $h(x) = \bar{g}^{-n}hg^n(x)$; if x is a source of g, $h(x)$ is defined as the source of $\bar{g}$ near x. It is easy to see that h is $1-1$ and onto. It remains to prove that h is continuous. It is clear that h is continuous on the stable manifolds of the sinks. The continuity of h on $W^s(p_2)$ follows from the fact that the restriction of h to N_2^u is continuous. To prove the continuity of h at the stable manifolds of the others saddle points and at the sources we proceed by induction on the phase diagram of g as in [5].

To finish the proof of Theorem 3.1 we have to prove the existence of the retraction π. Let $\bar{N}^s$ be a small neighborhood of $D_{\bar{g}}^s$. Denote by $\bar{N}_+^s$ the connected component of $\bar{N}^s$ containing $\bar{\Sigma}_j$. It is easy to construct a continuous fibration ρ: $\bar{N}_+^s \to \bar{\Sigma}_j$ satisfying the following properties: (a) in the coordinate system on U_2 linearizing $\bar{g}$, the fibers of ρ are graphs of piecewise linear functions; (b) if $x \in \bar{\Sigma}_j$ and $x_m = \rho^{-1}(x) \cap \bar{\Sigma}_m$ then $\pi_2^s(h^{-1}(x)) = \pi_s^2(h^{-1}(x_m))$; (c) if $y \in \rho^{-1}(x)$ is in the strip bounded by S_m^1 and S_m^2, $m = 1, \dots, j$, then $y = (0, y^2)$, i.e. the fibers of ρ coincide with the fibers of $\bar{\pi}_2^s$ in these strips.

Now let $\bar{N}_2^u = \bigcup_{n\geq 0} \bar{g}^n(\bar{N}^s)$. Let π: $\bar{N}_2^u \to W^u(\bar{p}_2)$ be defined by $\pi(x) = x$ if $x \in W^u(\bar{p}_2)$, $\pi(x) = \bar{\pi}_2^s(x)$, $\bar{g}^{-n}(x) \in \bar{N}^s - \bar{N}_+^s$ for some n and $\pi(x) = \bar{g}^n h\pi_2^s h^{-1}\rho\bar{g}^{-n}(x)$ if $\bar{g}^{-n}(x) \in \bar{N}_+^s$ for some n. It is easy to see that π satisfy the required properties. This finish the proof of Theorem 3.1.

§4. PROOF OF THEOREM D

Let m be the subset of $\mathrm{Diff}^\infty(M)$ defined in §1. We recall that if $f \in m$ then there is a unique pair of periodic saddle points of f, $p_1(f)$ and $p_2(f)$, such that one of the connected components of $W^u(p_1(f)) - \{p_1(f)\}$ is contained in $W^s(p_2(f))$. To simplify the notation we assume that $p_1(f)$ and $p_2(f)$ are fixed points and that the eigenvalues $\lambda_i(f)$ and $\mu_i(f)$ of $df(p_i(f))$ are positive with $\lambda_i(f) < 1$ and $\mu_i(f) > 1$ for $i = 1, 2$, and $f \in m$. The proof without this assumption is similar.

From p. 256 of [2], it follows that if the eigenvalues $\lambda_i(f)$ and $\mu_i(f)$ satisfy a certain number of non-resonance conditions then f is C^3-linearizable in neighborhoods of $p_i(f)$, $i = 1, 2$. Hence the set $\mathscr{m}^1 \subset \mathscr{m}$ of diffeomorphisms satisfying these conditions is open and dense in $\mathscr{m}$.

To each $f \in \mathscr{m}^1$ and $q \in W^u(p_1(f)) \cap W^s(p_2(f))$ we assign a C^2 function $T_{f,q}$: $[0, 1] \to \mathbb{R}$ which will be used in describing the conjugacy class of f. First we choose C^3 coordinate systems φ_i: $U_i \to \mathbb{R}^2$ linearizing f, where U_i is a neighborhood of $p_i(f)$, $i = 1, 2$, and U_1 contains q and $f(q)$. Let θ: $[0, 1] \to M$ be defined by $\varphi_1\theta(t) = \varphi_1(q) + t(\varphi_1(f(q)) - \varphi_1(q))$. Then θ is a C^3 embedding, $\theta(0) = q$, $\theta(1) = f(q)$ and $\theta([0, 1])$ is contained in $W^u(p_1(f)) \cap W^s(p_2(f))$. Now, let $T_{f,q}$: $[0, 1] \to \mathbb{R}$ be given by $T_{f,q}(t) = T_2(f, p_1, p_2, \theta(t), q)$ where T_2 is the conjugacy invariant defined in §2. We notice that $T_{f,q}$ is C^2 and the definition does not depend on the choice of the coordinate systems linearizing f.

LEMMA 4.1. *Let $\mathscr{n} \subset \mathscr{m}^1$ be a small neighborhood of $f \in \mathscr{m}^1$. Suppose $f' \in \mathscr{n}$ satisfy the conditions*: (i) $[(\log \mu_2(f'))/(\log \lambda_1(f'))] = [(\log \mu_2(f))/(\log \lambda_1(f))]$; (ii) *there is a homeomorphism ψ: $[0, 1] \to [0, 1]$ such that $\psi(0) = 0$, $\psi(1) = 1$ and $T_{f',q'}(\psi(t)) = T_{f,q}(t)$. Then f' is topologically equivalent to f. Furthermore, if $[(\log \mu_2(f))/(\log \lambda_1(f))]$ is irrational then the conditions (i) and (ii) are also necessary for the existence of a conjugacy h between f and f' such that $h(q) = q'$.*

Proof. To prove the necessity of the conditions (i) and (ii) we can use the same arguments of Lemma 2.5. The methods of §3 can be used to prove that the conditions (i) and (ii) are also sufficient for the existence of a conjugacy.

LEMMA 4.2. *Given $\epsilon > 0$ there are neighborhoods $\mathscr{n}$ of f and N of q such that if $f' \in \mathscr{n} \cap \mathscr{m}^1$ and $q' \in W^u(p_1(f')) \cap W^s(p_2(f')) \cap N$ then $T_{f',q'}$ is ϵC^2 near $T_{f,q}$.*

Proof: From the proof of Sternberg's theorem [8], we can choose neighborhoods U_i of $p_i(f)$ and, for each f' near f, C^3 coordinate systems $\varphi_{i,f'}$ on U_i linearizing f', such that $\varphi_{i,f'}$ depend continuously on f' in the C^3 topology. Therefore the lemma follows from the expression of T_2 in these coordinate systems (see §2).

Let $\mathscr{m}^2 \subset \mathscr{m}^1$ be the set of diffeomorphisms such that $f \in \mathscr{m}^2$ iff the critical points of $T_{f,q}$ are non-degenerate. By Lemma 4.2, $\mathscr{m}^2$ is open in $\mathscr{m}^1$. The lemma below is not difficult to prove.

LEMMA 4.3. *$\mathscr{m}^2$ is open and dense in $\mathscr{m}^1$.*

Let $\mathscr{m}_k \subset \mathscr{m}^2$ be such that $f \in \mathscr{m}_k$ if and only if the number of critical points of $T_{f,q}$ is equal to k-1, where $q \in W^u(p_1(f)) \cap W^s(p_2(f))$ is such that no critical point of $T_{f,g}$ occurs in the boundary of $[0, 1]$. By Lemma 4.2, $\mathscr{m}_k$ is open in $\mathscr{m}$. Furthermore, by Lemma 4.1, a diffeomorphism f' near f is topologically equivalent to f if $[(\log \mu_2(f'))/(\log \lambda_1(f'))] = [(\log \mu_2(f))/(\log \lambda_1(f))]$ and the critical values of $T_{f',q'}$ are equal to the corresponding critical values of $T_{f,q}$. Thus the modulus of stability of f is equal to k. This proves Theorem D.

REFERENCES

1. P. HARTMAN: On local homeomorphisms of Euclidean spaces, *Bol. Soc. Math. Mexicana* **5** (1960), (2).
2. P. HARTMAN: *Ordinary Differential Equations*. Wiley, New York (1964).
3. S. NEWHOUSE and J. PALIS: Bifurcations of Morse-Smale dynamical systems, *Dynamical Systems* (Edited by M. Peixoto). Academic Press, New York (1973).
4. J. PALIS: A differentiable invariant of topological conjugacies and moduli of stability, *Astérisque* **51** (1978).

5. J. PALIS: On Morse–Smale dynamical systems, *Topology* **8** (1969).
6. J. PALIS and S. SMALE: Structural stability theorems, global analysis, *Proc. Symp. in Pure Math. XIV*, *AMS* (1970).
7. C. ROBINSON: C^r structural stability implies Kupka–Smale, *Dynamical Systems* (Edited by M. Peixoto). Academic Press, New York (1973).
8. S. STERNBERG: On local structure of local homeomorphisms of Euclidean n-spaces II, *Am. J. Math.* **80** (1958).

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